Fundamental Principles of Just Intonation and Microtonal Composition

Thomas Nicholson and Marc Sabat

Universität der Künste Berlin
Studio für Intonationsforschung und mikrotonale Komposition

Contents

1. Tuning as a perceptual practice

Just intonation describes a particular practice of playing "in tune" – namely, of tuning musical intervals as small number frequency *ratios*[1] to evoke a distinctive periodic resonance. This *harmonic fusion* is perceived most clearly when hearing an aggregate of frequencies tuned according to an harmonic series. Such a sound more nearly represents the *spectrum of a single tone*.[2] Harmonic fusion is often perceived in naturally occurring acoustic structures composed of *harmonic partials*,[3] *i.e.* frequencies, which are whole number multiples of a single *fundamental* frequency.[4] Such sounds have a salient pitch and a periodic waveform. The characteristic *untempered* intervals and aggregates of harmonic series suggest ways of perceiving and exploring this fusion sound within musical contexts. These intervals include infinitely many that are *microtonal*.[5]

When two pitches play simultaneously, a listener perceives an interval – *i.e.* a specific *quality of sound*. Each pitch's timbre is an harmonic series and its perceived pitch-height is the series' fundamental. The pitches interact to produce combination tones.[6] As the interval between two fundamental frequencies approaches a simple ratio, some of their respective partials come into alignment. This highlights unisons between partials by slowing down or eliminating the sensation of *beating*[7] and, thereby, focusses the interval's characteristic sonority. At the same time, the composite waveform of the two pitches becomes periodic, and produces a virtual fundamental called the *periodicity pitch*. Such special relationships seem, therefore, to be "tuned" and possess a recognisable *periodic signature*. This perceived quality may be correlated to the unique pattern of frequency differences between the combined partials of both pitches, which repeats at multiples of the interval's *least common partial*.[8]

1. Frequencies are perceived proportionally, *i.e.* a constant ratio of two frequencies is heard as a constant musical interval.
2. See Section 7 for further discussion.
3. Partials are individual frequency components of a single sound. *Inharmonic* partials do not necessarily follow the harmonic series (*e.g.* the vibration of a drumhead or cymbal).
4. The harmonic partials of a fundamental frequency f are equal to f, $2f$, $3f$, $4f$, *etc.*
5. This term is generally applied to any interval smaller than a wholetone, which is not one of the familiar semitones.
6. Combination tones (summation tones and difference or Tartini tones) are frequencies caused by nonlinear interference between simultaneous vibrations in a medium. The combination tones of two frequencies f_1 and f_2 take the form $|(a \times f_1) + (b \times f_2)|$, where a and b are positive or negative integers.
7. As the frequency difference between two pitches approaches zero, pulsating changes of loudness called *acoustic beats* are perceived.
8. The least common partial is the lowest frequency that is an harmonic partial of both

Harmonic fusion and periodic signatures are produced by intervals that are part of an harmonic series and may, therefore, be represented by a ratio of whole numbers. For example, the interval of the octave-reduced natural seventh, *e.g.* 220 Hz to 385 Hz, is clearly discernible. The upper frequency relates to the lower in the same way that the seventh harmonic relates to the fourth harmonic – *i.e.* by the fraction $\frac{7}{4}$.

$$220\text{ Hz} \times \frac{7}{4} = 385\text{ Hz}$$

Similarly, to determine the harmonic relationship between two known frequencies (in this case 220 Hz and 385 Hz), they may be divided by their greatest common divisor or GCD (here 55), thereby reducing the ratio to lowest terms. This simplified ratio serves as the interval's most accurate identification and, at the same time, defines one pitch in terms of another.

$$\frac{385\text{ Hz} \div 55}{220\text{ Hz} \div 55} = \frac{7}{4}$$

2. The language of ratios

Composer Harry Partch (1901–1974) developed a method of working with pitches expressed as ratios,[9] measuring intervals from a single reference, written in the form $\frac{1}{1}$. The reference Partch chose for his scale was G3 392 Hz, although any frequency may be used as a reference. Since the pitch A4 is commonly used as a tuning reference for orchestral and chamber playing, it is perhaps the most convenient standard $\frac{1}{1}$ for instrumental just intonation composition. In the following examples, the fractions may be thought of as pitches in this sense.

To combine two pitches, their ratios are multiplied.

$$\frac{b}{a} \times \frac{d}{c}$$

To reduce the product to lowest terms, the numerator and denominator are each divided by the their greatest common divisor (GCD).

To find the *interval* between two pitches, the larger ratio is divided by the smaller and the result reduced.

$$\frac{b}{a} \div \frac{d}{c}$$

pitches.

9. In his book *Genesis of a Music*, Partch calls this system *Monophony* or the *language of ratios*.

To divide two fractions, the first is multiplied by the reciprocal of the second.

$$\frac{b}{a} \div \frac{d}{c} \equiv \frac{b}{a} \times \frac{c}{d}$$

To transpose a pitch up (or down) by *one* octave, its frequency is doubled (or halved). To transpose a pitch upward by any number of octaves y, its frequency is multiplied by 2^y. To transpose downward, its frequency is divided by 2^y. As above, this is equivalent to multiplying by the reciprocal $\frac{1}{2^y}$ or 2^{-y}.

As an example, to transpose the lowest A of the piano to the highest,[10] its frequency is multiplied by seven octaves (2^7).

$$27.5 \text{ Hz} \times 2^7 \equiv 27.5 \text{ Hz} \times 128 = 3520 \text{ Hz}$$

Intervals may be conceived of in two ways depending on which of the two pitches is taken as reference. If the lower pitch is the reference, the interval is written as a fraction greater than 1, *e.g.* $\frac{3}{2}$ (A4 in relation to D4). If the higher pitch is the reference, the interval is written as a fraction between 0 and 1, *e.g.* $\frac{2}{3}$ (D4 in relation to A4). Note that frequency ratios are always ratios of positive numbers.

Pitches as well as intervals may be expressed in the form of fractions $\frac{b}{a}$. When pitches are sounded successively, their interval may be called *melodic*. Melodic intervals and aggregates of three or more pitches (chords, melodies) are sometimes more conveniently expressed as a proportion.

$$a : b : c : ...$$

If an harmonic series is a fundamental frequency multiplied by whole numbers $\{1,2,3,...,n\}$, a *subharmonic* series is a common partial frequency divided by $\{1,2,3,...,n\}$. This is equivalent to multiplying by the reciprocals of the whole numbers $\{\frac{1}{1},\frac{1}{2},\frac{1}{3},...,\frac{1}{n}\}$. The result is an inverted harmonic series, which has the same sequence of intervals projected successively downward.

The subharmonic series' application to music is often criticised because it is not a "naturally occurring" psychoacoustic structure – *i.e.*, it is not a perceived phenomenon of harmonic auditory cognition like fusion, periodic signature, or timbre. It is, nevertheless, a useful musical model, as compositions of Ben Johnston (b. 1926) exemplify.[11]

10. This calculation ignores the common practice of tuning octaves on pianos slightly wider than the ratio 1:2 (stretch tuning). This accomodates the slight inharmonicity of the instrument's metal strings, which are actually too short and thick for the frequencies needed.

11. In particular, Johnston's string quartets, *e.g.* Nos. 5, 6, and 7, work with serial transformations (prime, retrograde, inversion, retrograde inversion). These are applied to melodic material tuned in just intonation, harmonised by harmonic or subharmonic series pitch-class

The various nodes of a single natural harmonic played upon an open string follow the subharmonic series downward. For example, the 7th partial may be played at nodes located at $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, and $\frac{6}{7}$ of the string length; these pitches, when stopped, produce a subharmonic series below the 7th partial. As well, the valves of brass instruments, when tuned proportionally to the main tube length and to each other, generate a subharmonic series of fundamentals above which the players can produce harmonic series.

Partch generally wrote pitches in "normalised" form – *i.e.* reduced to the octave between $\frac{1}{1}$ and $\frac{2}{1}$. Intervals greater than the octave are divided by the appropriate power of 2 and reduced to lowest terms. For example, the perfect eleventh $\frac{8}{3}$ exceeds the octave $\frac{2}{1}$ by a perfect fourth $\frac{4}{3}$. The following demonstrates this normalisation procedure.

$$\frac{8}{3} \div 2 \equiv \frac{8}{3} \times \frac{1}{2}$$

$$\frac{8}{3} \times \frac{1}{2} = \frac{8}{6}$$

$$\mathrm{GCD}(8,6) = 2$$

$$\frac{8 \div 2}{6 \div 2} = \frac{4}{3}$$

$$1 \leq \frac{4}{3} \leq 2$$

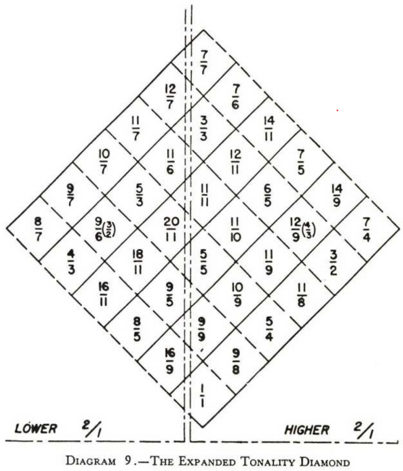
Partch defined harmonic series pitch aggregates ($\frac{1}{1}$,$\frac{2}{1}$,$\frac{3}{1}$,...) as *otonal* structures. Following the idea of harmonic dualism,[12] he defined parallel subharmonic series pitch aggregates ($\frac{1}{1}$,$\frac{1}{2}$,$\frac{1}{3}$,...) as *utonal* structures. Distinct[13] normalised otonal and utonal ratios up to the 11th partial form six-note constellations that he called hexads –

subsets. These transformations translate between harmonic and subharmonic structures.

12. Harmonic dualism is an attempt to symmetrically explain major/minor tonality, as expressed in theoretical works by Rameau, Tartini, Oettingen, Riemann, *et al.*

13. Since even number partials are octave transpositions of lower partials, only the odd partials produce new pitch classes.

Figure 1: Harry Partch's tonality diamond, which also served as the layout of his Diamond Marimba.



$$\frac{1}{1} : \frac{3}{2} : \frac{5}{4} : \frac{7}{4} : \frac{9}{8} : \frac{11}{8} \equiv \text{otonal}$$

and

$$\frac{1}{1} : \frac{4}{3} : \frac{8}{5} : \frac{8}{7} : \frac{16}{9} : \frac{16}{11} \equiv \text{utonal}$$

– interlocked to form a "tonality diamond". With this construction, based on a model devised by Max F. Meyer (1873–1967), Partch invented a just intonation tone system and conceived of a musical instrument using this layout, which he called the Diamond Marimba.

3. Melodic distance

It is useful to have a method of comparing the "absolute sizes" of various intervals, or their *melodic distances* from $\frac{1}{1}$. Given two intervals written as fractions, it is not immediately clear which one is lesser or greater, nor to what extent, since their difference is only determined by dividing their ratios.

The proportional comparison of intervals or any perceived phenomenon is described in the science of psychophysics by two principles called Weber's Law and Fechner's Law, defined by Gustav Theodor Fechner (1801–1887). They were first published in his book *Elemente der Psychophysik*, which established the interdisciplinary study of how humans perceive the relative degree of physical magnitudes according to a quasi-logarithmic scale. He formulated the concept of *Just Noticeable Difference* or JND, which refers to the smallest change in a stimulus that may be perceived.

A logarithmic scale of finely-grained equal divisions may provide a “ruler” against which all intervals may be measured. Octave transpositions of a frequency have the unique quality of often being perceived as the same “note”,[14] so the simplest approach is to base this scale of measurement on equal divisions of the octave $\frac{2}{1}$, though this is to some extent an arbitrary choice.[15]

Since intervals are compounded by multiplication of their ratios, an equal division of the octave into n parts, expressed as a ratio, is the nth root of 2, where n is the number of divisions.

$$(\sqrt[n]{2})^n = \frac{2}{1}$$

Most equal-division intervals are *irrational* because they are expressed by means of radicals and may not be reduced to simple *whole number* fractions.[16]

Pitches that are irrationally related are not tuned in just intonation. Therefore, their composite waveform is not periodic and there is no common fundamental. A geometric progression is, however, useful for comparing intervals as its spacing is perceived as smooth and even.

12-tone equal temperament[17] is an example of a scale of equal divisions. Each step, called an equal-tempered semitone, is equivalent to the 12th-root of 2, which may also be expressed as 2 raised to the power of $\frac{1}{12}$.

$$\sqrt[12]{2} \quad \text{or} \quad 2^{\frac{1}{12}}$$

The ratio R_n of an equal-tempered interval comprised of n semitones is thus the ratio of one semitone, raised to the power n.

$$R_n = (2^{\frac{1}{12}})^n = 2^{\frac{n}{12}}$$

For instance, the wholetone is comprised of two semitones.

$$(2^{\frac{n}{12}})^2 = 2^{\frac{1}{6}}$$

14. A property called *octave equivalence*.

15. Other tunings divide different intervals. The Bohlen-Pierce scale, for example, comprises 13 equal divisions of the perfect twelfth ($\frac{3}{1}$).

16. An exception would be an interval that is an integer power of a fraction, *e.g.* the interval $\frac{64}{27}$, which is $\left(\frac{4}{3}\right)^3$, may be divided into 3 parts, each being $\frac{4}{3}$.

17. An equal temperament may be referred to as an ED2 (or EDO) – equal division of the octave. In this paper, “equal-tempered” refers to 12-ED2 unless otherwise specified. See Section 9 for further discussion.

Equivalently, for any ratio R, the number of comprised semitones $n_{semitones}$ may be calculated. For any interval outside the 12-ED2 gamut, this value will not be a whole number.

$$12 \times \log_2(R) = n_{semitones} \tag{1}$$

Mathematician Alexander J. Ellis (1814–1890) proposed the division of each semitone into very small, equal units of measure, *i.e.* to the extent that the human ear would be, for the most part, insensible to slight deviations from exact ratios.[18] By dividing each semitone into 100 units called *cents*, the octave is pixelated into an equal division scale of 1200 parts, each equivalent to the 1200th-root of 2. The general formula for a ratio's size in cents n_{cents} results if 1200 is substituted for 12 in the previous expression.

$$1200 \times \log_2(R) = n_{cents} \tag{2}$$

For most listeners, the JND is less than 10 cents,[19] though this depends on frequency, amplitude, and timbre, as well as previous experience. JND also becomes smaller when heard in a harmonic context, as many intervals – especially small number ratios such as the octave and the fifth – are particularly sensitive to tuning deviations, which are manifested by acoustical beats caused by slightly mistuned partials and/or combination tones.

Various other systems of logarithmic measure for musical intervals have been proposed. These include Joseph Sauveur's *mérides* (43-ED2), closely related to the $\frac{1}{5}$-comma meantone temperament used in France at the time, which he further subdivided into *eptamérides* (301-ED2) and *decamérides* (3010-ED2).[20] Arthur von Oettingen (1836–1920), following a proposal by English scientist Sir John Herschel (1792–1871), used *millioctaves* (1000-ED2). The MIDI standard defined various microtonal pitch deviations applicable to digitally controlled musical instruments – *e.g.*, the 14-bit pitch bend value can be applied to an arbitrary interval measured in 12-ED2, dividing it into 16384 equal parts.[21]

18. Hermann von Helmholtz, *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, Second English Edition, trans. Alexander J. Ellis (New York: Dover, 1954), p. 431.

19. The authors can attest to melodic discrimination as fine as 2 cents in the register around 260 Hz (middle C4) and harmonic discrimination < 0.1 cent with computer generated harmonic spectra or three-tone sinewave chords presented in the ratio $f_1 : (f_1 + f_2)/2 : f_2$ by matching difference tones.

20. Joseph Sauveur (1653–1716) was a French mathematician, physicist and a founder of the science of musical acoustics.

21. midi.org, *The MIDI 1.0 Specification*, https://www.midi.org/specifications-old/item/the-midi-1-0-specification, 1982.

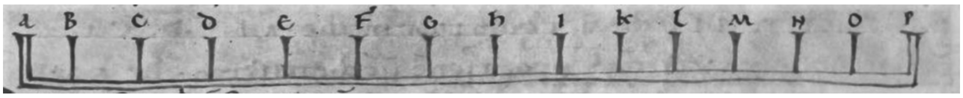
If the octave is divided into a whole number of equal units, then they *cannot* be whole number ratios. Conversely, if an octave is divided into just intonation intervals, they *must* be unequal and incommensurate. In this sense, the particular complexities of just intonation and equal temperaments are inversely related to each other.[22]

4. Microtonal notations

The Western five-line staff notation is fundamentally Pythagorean[23] and diatonic. The diatonic notes A through G divide each octave into five wholetones and two diatonic semitones.

The ancient Greek *Greater Perfect System* begins, from lowest to highest note, with a wholetone followed by two conjunct tetrachords. Each rising tetrachord, when tuned diatonically,[24] consists of a diatonic semitone followed by two wholetones. This system was represented in *De institutione musica* by Anicius Manlius Severinus Boethius (477–524) in the form of a diagram using the successive letters A, B, C, D, E, F, G, *etc.* (Figure 2), which came to be used as diatonic note names. The distinction between the conjoined and disjoined third tetrachords *synemmenon* and *diezeugmenon* in the second octave was notated using two forms of B – *molle*, written as ♭ and *durum*, written as ♮ (eventually becoming B♭ and B♮). These eight notes comprise the *musica vera* gamut.

Figure 2: Boethius' labelling in *De institutione musica* of the diatonic notes as a progression of letters.



Transpositions of the diatonic semitones B–C and E–F are written by means of the additional "accidentals" ♭ and ♯, the second of which was introduced by Marchetto da Padova in the 1300s: *e.g.* A–B♭ and F♯–G.[25] The naming of intervals – unison, second, third, *etc.*, and their "*enharmonic* alter-

22. An in-depth discussion of this topic may be found in Easley Blackwood's book *The Structure of Recognizable Diatonic Tunings*. For further analysis of equal-division tone systems see Section 9.

23. "Pythagorean" refers to intervals combining only the primes 2 and 3.

24. In Greek theory, the tetrachords are divided into diatonic, chromatic, and enharmonic divisions of the perfect fourth and are generally given in descending order of pitch.

25. Karol Berger, *Musica ficta: Theories of accidental inflections in vocal polyphony from Marchetto da Padova to Gioseffo Zarlino* (Cambridge: Cambridge University Press, 2004), p. 22.

ations", *i.e.* diminished and augmented intervals – are based on this notation. The structure is Pythagorean since B♭ and F♯ correct the intervals at both ends of the diatonic chain to continue the series of perfect fifths indefinitely in both directions.

(B♭) – F – C – G – D – A – E – B – (F♯)

Over the course of several hundred years, the Pythagorean consonances (octave, fourth, fifth) came to be complemented in the practice of vocal music by various *imperfect* consonances (major/minor thirds and sixths sung as small number ratios).

In *Harmonics*, Claudius Ptolemy (*ca.* 100–170) provided string lengths for an entire octave tuned in his *tense diatonic* "genus" – 60, $66\frac{2}{3}$, 75, 80, 90, 100, $112\frac{1}{2}$, 120.[26] These correspond to a descending scale with the following melodic frequency ratios – 10:9, 9:8, 16:15, 9:8, 10:9, 9:8, 16:15. Whenever two wholetones occur in succession, they are of two different sizes and together comprise the ratio $\frac{5}{4}$, called the Ptolemaic major third.[27] Tuned as Ptolemaic intervals, thirds and sixths differ from their dissonant Pythagorean counterparts by the interval $\frac{81}{80}$, known as the *syntonic comma* ($\kappa_5 = 21.51$ cents).[28]

As Ptolemaic intervals entered into common practice among musicians, they gradually became accepted by theorists in descriptions of monochord tunings.[29] Gioseffo Zarlino (1517–1590) is credited with introducing the Ptolemaic tense diatonic as a basis for music theory. He expanded the Pythagorean definition of consonances to the *senario*, which comprised proportions drawn from the numbers 1, 2, 3, 4, 5, 6, and 8. In particular, he described the two different melodic divisions of the Ptolemaic major third, although he did not take the further step of introducing an explicit notation of this difference.

Experimental keyboards tuned in just intonation were explored by Zarlino and, among others, Francisco de Salinas (1513–1590),[30] but the necessity for

26. Andrew Barker, *Greek Musical Writings, Volume II: Harmonic and Acoustic Theory* (Cambridge: Cambridge University Press, 1997), p. 350.
27. "Ptolemaic" refers to intervals combining only the primes 2, 3, and 5.
28. In this article, the Greek letter κ stands for comma (*cf.* Section 5) and the numerical subscript refers to its relevant prime dimension. The "main" commas of each dimension (*e.g.* the Pythagorean comma, the syntonic comma, the septimal comma, *etc.*) have been given this numerical notation (*i.e.* κ_3, κ_5, κ_7, *etc.*) while other commas are assigned subscripts that relate to their traditional naming (*e.g.* skhisma = κ_{sk}).
29. Bartolomé Ramos de Pareja, *Musica practica* (?Bologna: Baltasar de Hiriberia, 1482), p. 5.
30. Patrizio Barbieri, *Enharmonic Instruments and Music 1470–1900* (Rome: Il Levante Libreria Editrice, 2008), pp. 30–33.

additional keys and complex multi-rank layouts inhibited the general adoption of such instruments. Instead, various keyboard temperaments arose to better approximate these new sounds.[31] The so-called *sistema partecipato* (meantone system), which divides the syntonic comma geometrically in four equal parts, became standard practice. Each perfect fifth is deliberately narrowed by $\frac{1}{4}\kappa_5$ so that four successive fifths produce a Ptolemaic, rather than Pythagorean, major third. These new pitches may still be represented using conventional notation since the standard meantone tuning encompasses only twelve notes (the series of fifths from E♭ through G♯). On the other hand, extended meantone instruments with split black keys distinguishing the difference of a *lesser diesis* $\frac{128}{125}$ ($\kappa_{ld} = 41.06$ cents) between sharps and flats,[32] *e.g.* the *cembalo cromatico*, did also achieve a certain measure of success in Italy and, though rare, continue to be built today.

The Pythagorean wholetone is the difference between the perfect fifth and the perfect fourth.

$$\frac{3}{2} \div \frac{4}{3} = \frac{9}{8}$$

Two successive Pythagorean wholetones produce the Pythagorean ditone (or major third).

$$\frac{9}{8} \times \frac{9}{8} = \frac{81}{64}$$

$$1200 \times \log_2\left(\frac{81}{64}\right) = 407.82 \text{ cents}$$

If A is tuned to 0 cents, this Pythagorean F a ditone below will be tuned -8 cents in relation to equal-tempered F.[33]

$$1200 \times \log_2\left(\sqrt[3]{2}\right) = 400 \text{ cents}$$

The Pythagorean ditone is larger than the Ptolemaic major third by a syntonic comma.

$$\frac{81}{64} \div \frac{5}{4} = \frac{81}{80}$$

31. A complete discussion of the historical development of organ and other keyboard temperaments falls outside the scope of this article. Excellent discussions of the topic may be found in J. Murray Barbour's *Tuning and temperament* and Klaus Lang's *Auf Wohlklangswellen durch der Töne Meer.*

32. See Section 5.

33. Cents are usually indicated on an electronic tuner in relation to 12-ED2.

The Ptolemaic F below A will, therefore, be tuned 22 cents higher than the Pythagorean F and 14 cents higher than equal-tempered F.

$$1200 \times \log_2 \left(\frac{5}{4} \right) = 386.31 \text{ cents}$$

Another way of considering the implications of this comma is to observe that the Ptolemaic major third is comprised of of two different intervals, a *major wholetone* and a *minor wholetone*.

$$\frac{9}{8} \times \frac{10}{9} = \frac{5}{4}$$

Meantone systems compromise the difference between these two wholetones by establishing a single irrational *mean*tone with ratio $\sqrt{\frac{5}{4}}$.

In order to clarify the relationship between interval and notation, many special accidental systems have been devised. Nicola Vicentino (1511–1575) notated the lesser diesis with a dot above the note and a $\frac{1}{4}\kappa_5$ alteration with a comma (Figure 3) to differentiate two different tuning systems he proposed for his *archicembalo*.[34] Zarlino suggested $\times$ as a symbol for the lesser diesis (Figure 4) in *Le istitutioni harmoniche* (1558). Vicente Lusitano (d. after 1561) divided the wholetone in 9 "commas" and notated them with various numbers of strokes to show different amounts of sharpening[35] (Figure 5). Giuseppe Tartini (1692–1770) introduced a new symbol for the natural seventh $\frac{7}{4}$ (Figure 6 and Figure 7) in his *Trattato di musica* (1754).

A century later, Oettingen, Moritz Hauptmann (1792–1868), as well as Hermann von Helmholtz (1821–1894) advocated for the explicit notation of the syntonic comma to pursue an adoption of Ptolemaic just intonation rather than the 12-ED2 system, which was gaining popularity through the industrial production and distribution of pianos.

In the beginning of the twentieth century, Partch devised several systems of accidentals before finally adopting a ratio-based tablature notation for his instruments. The early twentieth century also saw the emergence of various ways of notating equal divisions of a tone. These include the well-known quartertone symbols that were introduced by Richard Heinrich Stein (1882–1942) as well as the accidental systems of Ivan Wyschnegradsky (1893–1979) (Figure 8), Alois Hába (1893–1973), and numerous other twentieth century composers. Especially unique is the compact one-line notation of Julián Car-

34. Nicola Vicentino, *L'antica musica ridotta alla moderna prattica* (Venice: Antonio Barre, 1555).

35. Vicente Lusitano, *Introduttione facilissima, et novissima, di canto fermo, figurato, contraponto semplice, et in concerto* (Rome: Antonio Blado, 1553).

rillo (1875–1965) for *any* equal-division system, representing pitch classes by ordinal numbers (Figure 9).

It has become common among microtonal composers and theorists who do not wish to use a microtonal notation to notate the closest 12-ED2 analogue of a pitch and include an indication in cents of its deviation. In some cases, however, this approach suggests enharmonic[36] substitutions based on the logic of 12-ED2 (and derivates thereof, like 24-ED2 and 72-ED2) that falsify intervallic relations implied by traditional notation.

More recently, composers are developing and using notations that encode an interval's information *symbolically*, which may optionally be combined with cent deviation indications. Examples include Ben Johnston's notation, the *Extended Helmholtz-Ellis JI Pitch Notation (HEJI)* by Marc Sabat (b. 1965) and Wolfgang von Schweinitz (b. 1953), as well as *Sagittal Notation* by George Secor (b. 1943) and David Keenan (b. 1959).[37]

In the remainder of this article, musical examples of just intonation are notated using *HEJI* (see Table 1 in the next section for a legend of the symbols).

Figure 3: Vicentino's dot and comma notation demonstrating the almost just $\frac{8}{7}$ septimal wholetone and the just $\frac{6}{5}$ minor third that results from raising the meantone minor third by $\frac{1}{4}\kappa_5$.

36. See Section 5.

37. Information on *HEJI Notation* can be found at `www.marcsabat.com` and information on *Sagittal Notation* can be found at `www.sagittal.org`. Information about Ben Johnston's notation can also be found on Marc Sabat's website in an article titled "On Ben Johnston's Notation and the Performance Practice of Extended Just Intonation" (2009).

Figure 4: Zarlino's notation of the chromatic and enharmonic tetrachords using crosses.

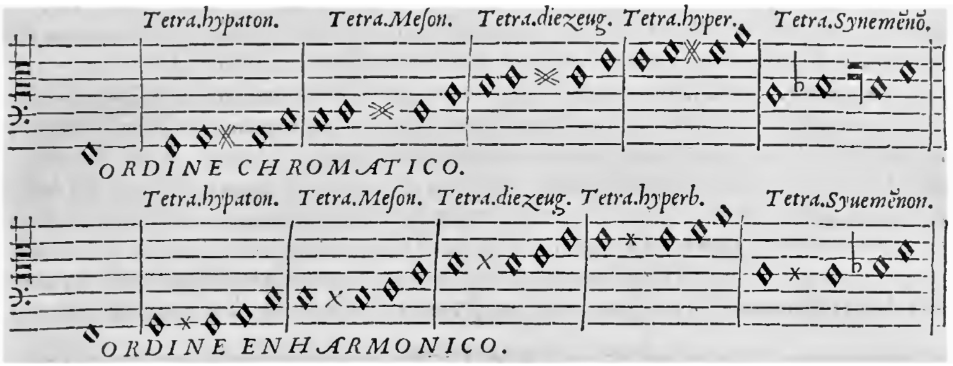


Figure 5: Lusitano's cross and stroke notation. Each stroke represents one "comma". The engraved example is a Ptolemaic interpretation of Lusitano's harmonisation of the melodic diesis.

Figure 6: Tartini notated the natural seventh with an symbol that looks like an inverted “7”.

Figure 7: Tartini composed figured bass examples demonstrating the septimal enharmonic mode and the natural seventh treated as a consonance, melodically rising.

Figure 8: Excerpt from Wyschnegradsky’s *Ainsi parlait Zarathoustra* (1930).

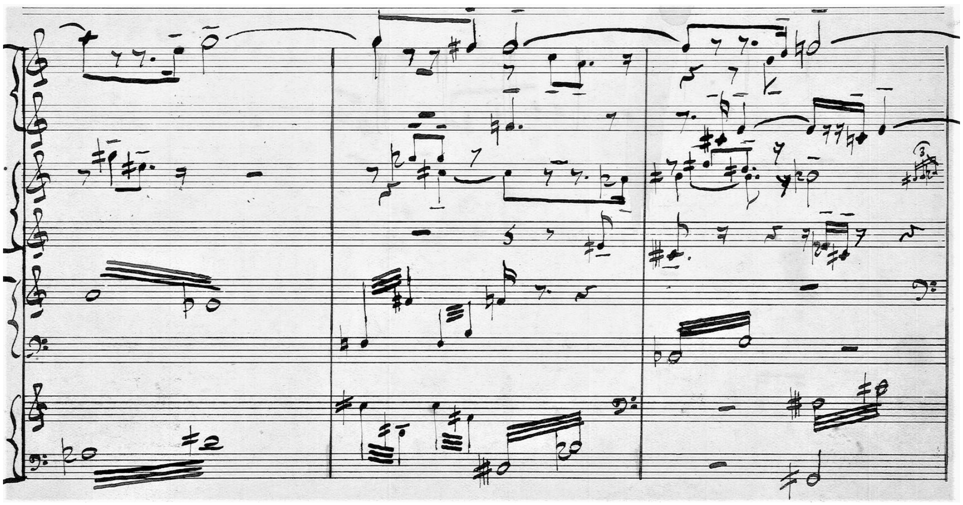


Figure 9: Carrillo used a tablature notation for microtonal intervals, as in *Preludio a Cristobal Colón* (1922) shown here.

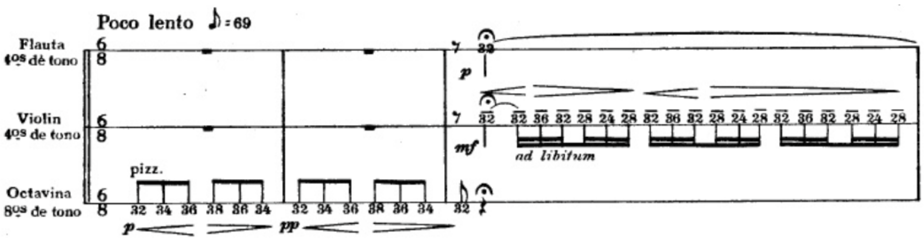
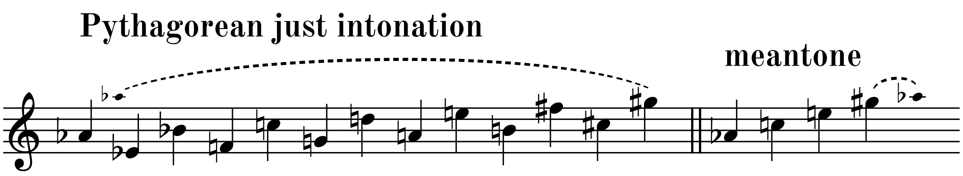


Figure 10: Notation of (a) the Pythagorean major third [or ditone] and (b) the Ptolemaic major third between the notes A and F in various just intonation accidental systems.

5. Commas and enharmonics

In a strict Pythagorean interpretation of the conventional staff notation, the *enharmonic* interval of a diminished second between A♭–G♯ or any transposition thereof is called the *Pythagorean comma* $\frac{531441}{524288}$ ($\kappa_3 = 23.46$ cents). A♭ is one comma *lower* than G♯. In $\frac{1}{4}\kappa_5$ meantone temperament the ratio $\frac{5}{4}$ is just and, consequently, the *same* difference of spelling is called the lesser diesis. Note that, in this case, A♭ is one diesis *higher* than G♯.

Figure 11: The fact that the same notation has been used historically in the two ways shown below has led to commonly held uncertainties about the contextual intonation of flats and sharps.



Staff notation does not normally differentiate between these two visually identical representations of two different enharmonic intervals because it is fundamentally one-dimensional. It may be understood to represent intervals based on multiples of the prime number 3 *or* intervals based on multiples of the prime number 5, but not combinations of both. To accurately depict multi-dimensional harmonic space, individual notes of the Pythagorean series of fifths, based on prime number 3, must be altered by various explicitly notated *commas* to represent interval ratios based on primes 5, 7, *etc.* and their combinations – *i.e.* additional symbols must be introduced.

Enharmonics are defined as differences of *spelling*, which may or may not involve (small) differences of intonation, depending on the tone system used. *Commas* are defined as (small) differences of *intonation*, occurring between enharmonics *or* between different microtonal variants of a single note.

Enharmonic differences do not necessarily constitute differences of intonation (*e.g.* E and F♭ in 12-ED2). In just intonation, however, enharmonic differences of spelling are *always* separated by the interval of some comma.

Figure 12: Lesser diesis between G♯ and A♭ around Pythagorean C♮ notated in just intonation. Each arrow represents a raising or lowering of the Pythagorean notes by one syntonic comma. Note that the Ptolemaic *diatonic semitone* $\frac{16}{15}$ (c) comprises the *minor chroma* $\frac{25}{24}$ (a) and the *lesser diesis* $\frac{128}{125}$ (b). See the Lusitano example in Section 4 for a contrapuntal setting of these microtonal intervals.

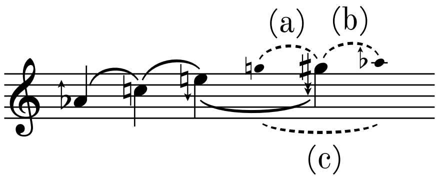


Table 1 presents comma notations introduced by *HEJI* for primes up to 31. Most alterations affect the basic Pythagorean pitch classes. Note, however, that both κ_{17} and κ_{29} affect the 5-dimension to notate the 17th and 29th partials respectively, modifying the commonly occurring Ptolemaic ratios $\frac{16}{15}$ and $\frac{9}{5}$. In addition, κ_{31} is applied to the 11-dimension to notate the 31st partial, modifying the undecimal quartertone $\frac{33}{32}$. Refer to Figure 16 in the next section to see how these accidentals are used to notate the first 32 partials of the harmonic series.

Since each prime requires its own comma, combining primes compounds their respective comma notations. For instance, the 35th partial (5×7) is lowered by two commas – syntonic (κ_5) and septimal $\frac{64}{63}$ ($\kappa_7 = 27.26$ cents).

$$\frac{81}{80} \times \frac{64}{63} = \frac{36}{35}$$

Table 1: HEJI Notation.

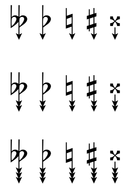
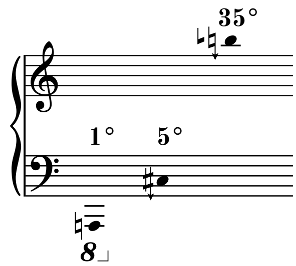
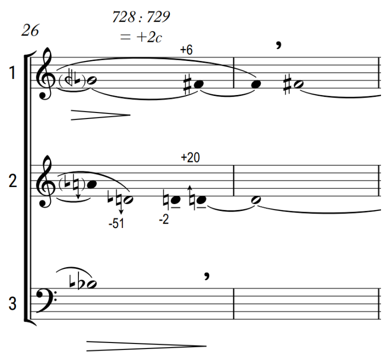
Prime	Otonal Notation	Utonal Notation	Comma	Deviation (cents)
3	𝄫 ♭ ♮ ♯ 𝄪		*lowers / raises by* Pythagorean apotome $(\frac{2187}{2048})$	113.69
5			*lowers / raises by* syntonic comma $(\kappa_5 = \frac{81}{80})$	21.51
7			*lowers / raises by* septimal comma $(\kappa_7 = \frac{64}{63})$	27.26
11			*raises / lowers by* undecimal $\frac{1}{4}$-tone $(\kappa_{11} = \frac{33}{32})$	53.27
13			*lowers / raises by* tridecimal $\frac{1}{3}$-tone $(\kappa_{13} = \frac{27}{26})$	65.34
17			*lowers / raises by* 17-limit skhisma $(\kappa_{17} = \frac{256}{255})$	6.78
19			*raises / lowers by* 19-limit skhisma $(\kappa_{19} = \frac{513}{512})$	3.38
23			*raises / lowers by* 23-limit comma $(\kappa_{23} = \frac{736}{729})$	16.54
29			*raises / lowers by* 29-limit comma $(\kappa_{29} = \frac{145}{144})$	11.98
31			*lowers / raises by* 31-limit skhisma $(\kappa_{31} = \frac{1024}{1023})$	1.69

Figure 13: Notation of the 35th partial – the 7th partial of the 5th partial, or, equivalently, the 5th partial of the 7th partial – of the lowest note on the piano (A0).



Xenharmonic Wiki[38] lists nearly 100 named commas ranging from 3.5 to 100 cents. There are as many commas as there are ways to tune any given interval. Because of this, JI notation can sometimes become unwieldy to read, *e.g.* when using more than three symbols in *HEJI*. One approach is simply to avoid such situations by limiting the harmonic space. Another possibility is to make an *enharmonic leap*, joining two points that nearly coincide. This may simplify the spelling by means of a small pitch jump and thereby facilitate a recentering of the harmonic space.

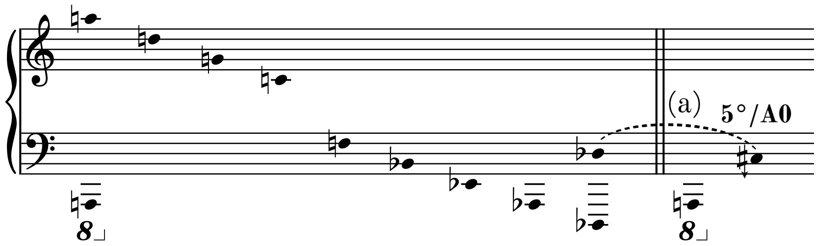
Figure 14: Excerpt from *BRANCH: Plainsound Trio* by Thomas Nicholson.



A movement through eight descending Pythagorean perfect fifths from the note A reaching D♭ differs by one skhisma ($\kappa_{sk} = \frac{531441}{524288} = 1.95$ cents) from the C♯ tuned as the fifth partial of the original note A (see Figure 15). This very small interval may serve as a useful "connection" between Pythagorean and Ptolemaic as will be discussed in the following section.

38. `http://en.xen.wiki`

Figure 15: Construction of the skhisma (a) between the notes D♭ and C♯.



6. Prime limits

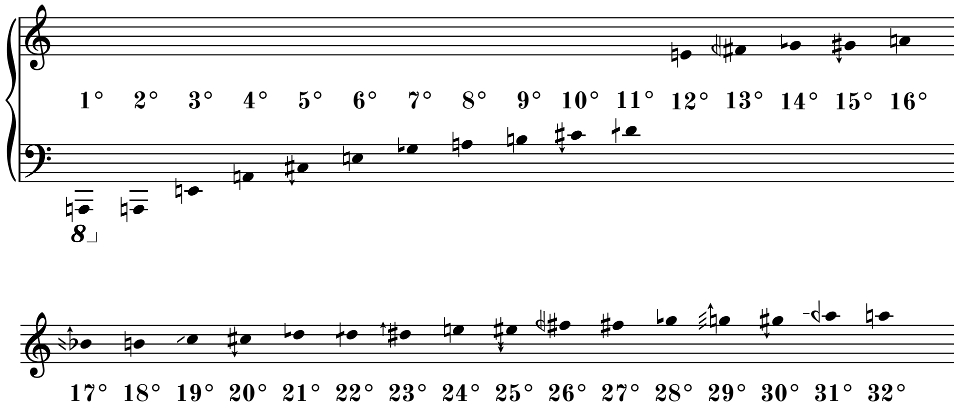
Pieces of music are traditionally said to share a common *form* when they have similar large-scale temporal characteristics (*e.g.*, minuet, rondo, sonata, moment-form, *etc.*). Analogous categorisations have been made with regard to the local temporal characteristics denoted by proportional *rhythm*[39] (*e.g.*, mensuration, hemiola, polyrhythm, metric modulation *etc.*) and may also be applied to the "micro-temporal" properties of tuning structures.[40] Partch introduced the concept of *prime limit* – the largest prime number used to generate intervals comprising a given tone system.

Prime limit categorisation allows a listener to know something about the type and degree of tuning complexity of a piece of music and how the tuning compares to that of other pieces. The specific prime numbers in a ratio determine its sonority because each prime generates a distinctive new interval. It is, therefore, useful to return to the harmonic series to consider each prime's intervals in relation to the other partials.

39. Henry Cowell, *New Musical Resources* (New York: Alfred A. Knopf, 1930), Part II: Rhythm.

40. Ben Johnston, *Maximum Clarity and Other Writings on Music*, ed. Bob Gilmore (Chicago: University of Illinois Press, 2006), "Scalar Order as a Compositional Resouce" (1965).

Figure 16: The first 32 partials of the harmonic series of A0 notated in *HEJI*.



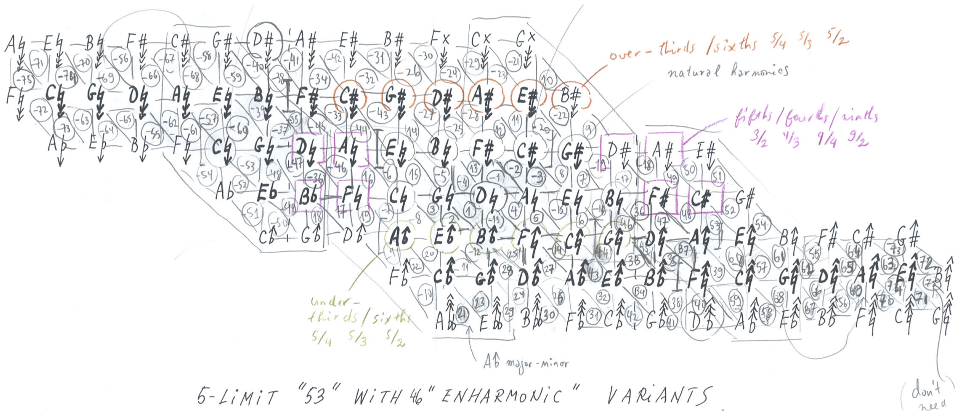
The sound of intervals related as multiples or powers of the same prime factor(s) share common characteristics and intonation. For example, the 15th partial combines with the 5th partial and the 3rd partial to create a minor triad 10:12:15 that complements the major triad 4:5:6 by inverting the order of the two Ptolemaic thirds $\frac{5}{4}$ and $\frac{6}{5}$. Prime limit informs a listener about the constrained collection of ways intervals may be tuned in a given context.

2-limit just intonation only allows for octave transpositions of a single tone. 3-limit (Pythagorean) just intonation additionally allows for the perfect fifth and its inversion[41] (*i.e.* the perfect fourth) as well as their powers (stacks of fifths and fourths), filling in the frequency range with *different notes*. 5-limit (Ptolemaic) just intonation supplements the Pythagorean pitch space with pure major and minor thirds and their inversions (*i.e.* pure major and minor sixths). Each step in the Ptolemaic dimension generates a new set of Pythagorean pitches offset by a syntonic comma.[42]

To better visualise this concept, it is possible to draw a *Tonnetz* or lattice diagram in the manner devised by Leonhard Euler (1707–1783).

41. The inversion of an interval $\frac{b}{a}$ is the octave complement $2 \div \frac{b}{a} \equiv \frac{2a}{b}$.
42. Every new prime number generates its own dimension by combining with each of the previous primes to replicate the lower prime limit harmonic space offset by a new comma.

Figure 17: 5-limit lattice diagram from the string quartet *Euler Lattice Spirals Scenery* (2011) by Marc Sabat.



As defined previously in Figure 15, an enharmonic connection exists between the 3- and 5-limit that allows for a good approximation of consonant 5-limit triads by means of exclusively 3-limit ratios with a small error equal to κ_{sk}. In Ellis' description of this construction, which he calls *skhismatic temperament*,[43] he extends the Pythagorean series, stating that

> [t]he condition is that the Fifths should be perfect and the Skhisma should be disregarded.
>
> ... Having an English Concertina (which has 14 notes) tuned in perfect Fifths from G♭ to C♯ ..., I have been able to verify ... that, although A–C♯–E [and] E–G♯–B are horrible chords, A–D♭–E [and] E–A♭–B are quite smooth and pleasant.

Ellis' "temperament", which is actually an enharmonic extension of Pythagorean intonation, may be constructed by tuning the perfect fifth C–G as 2:3 and projecting a series of six perfect fifths upward from G as well as six perfect fifths downward from C. By extending this system to ten fifths upward from G and ten downward from C, one arrives at the tuning of the 22 Indian *śrutis* proposed by Pichu Sambamoorthi (1901–1973).[44]

The term just intonation commonly refers to the 5-limit. Johnston uses the term *extended* just intonation[45] to refer to the inclusion of primes greater than 5. From these prime limits onward (7, 11, 13, *etc.*), timbral nuances and the

43. Helmholtz, *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, p. 435.

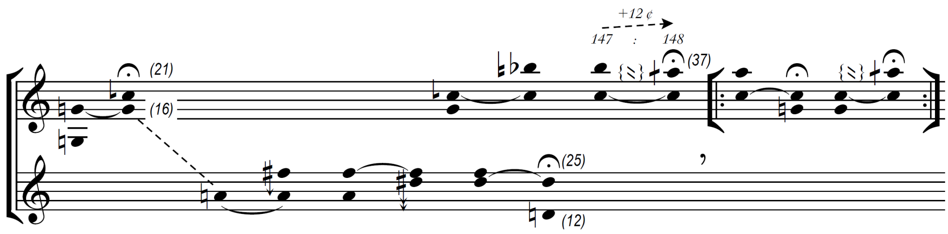
44. *cf.* Wolfgang von Schweinitz's transcription of Sambamoorthi's text and diagrams at `http://www.plainsound.org/pdfs/srutis.pdf`.

45. Johnston, *Maximum Clarity and Other Writings on Music*, p. 203.

possibilities for new intervals and chords become compositionally boundless. The 7-limit has a strong fusion sonority and remains, to a certain degree, intuitive, as the natural seventh $\frac{7}{4}$ is often used by Barbershop quartets and, occasionally, by brass instruments to tune dominant seventh chords. As well, the 7-limit may be heard in jazz music as an intonation of melodic "blue notes". In his innovative and ongoing work *The Well-Tuned Piano*, La Monte Young (b. 1935) uses primes 2, 3, and 7, but excludes 5 because of its historical connotations.[46] 11- and 13-limit tunings[47] open up the worlds of precisely tuned quarter- and thirdtones, which, to date, have primarily been explored in *melodic* contexts, *e.g.* in Arabic and Persian modal traditions.

Many composers have been interested in exploring the degree to which prime limit may be extended before harmonic nuances become effectively indistinguishable. Naturally, as with most things dealing with perception, there is no absolute answer because hearing is increasingly subjective as sounds become more complex. The upper bound for prime limit is probably between 23 and 31 for discerning (and trained) listeners.[48] Nevertheless, it is possible for higher primes to be tuned in aggregates when they are summation tones of lower sounds.[49]

Figure 18: 37 as summation tone of 12 and 25 from *Les Duresses* (2004) by Marc Sabat.



7. Consonance and dissonance

The terms *consonance* and *dissonance* are often employed to describe the interaction of tones. The words themselves very easily elude clear definition

46. To specify a particular subset within a prime limit, Tenney's concept of *projection space*, which collapses the 2-dimension to produce pitch-classes, may be used to differentiate between the general 7-limit (3,5,7-space) and Young's 7-limit (3,7-space); James Tenney, *From Scratch, Writings in Music Theory*, ed. Larry Polansky et al. (Illinois: University of Illinois Press, 2015), Chapter 18.

47. Partch's 43-tone scale is an 11-limit system.

48. See Section 7 for a discussion of tuneable intervals.

49. As in the *Fourth String Quartet – infinite to be cannot be infinite, infinite anti-be could be infinite* (1976–87) for 9 string quartets by Horațiu Rădulescu.

and have historically inspired many contradictory descriptions proposed by musicians, theorists, and scientists. There are two broad categories that may be distinguished. *Musical* consonance and dissonance refers to various, and shifting, systems of categories referring to sound-combinations as defined by different musical practices (dissonances requiring special context, preparation, resolution *etc*.). *Psychoacoustic* consonance and dissonance refers to perceived qualities of the composite sound like smoothness or roughness, harmonicity or inharmonicity, fusion, spectral balance, periodicity, and beating.

In his 1987 book *A History of Consonance and Dissonance*, composer James Tenney (1934–2006) identified five distinct consonance-dissonance concepts (CDCs) arising in Western music culture.

> ... the words *consonance* and *dissonance* ... have been used, historically, in at least *five different ways* – expressing five distinctly different forms of the CDC. Before the rise of polyphonic practice they were used in an essentially melodic sense, to distinguish degrees of affinity, agreement, similarity, or *relatedness between pitches* sounding successively. During the first four centuries of the development of polyphony they were used to describe an aspect of the *sonorous character* of simultaneous dyads, relatively independent of any musical context in which they might occur. In the 14th century the CDC began to change (again) in conjunction with the newly developing rules of counterpoint, and a new system of interval-classification emerged which involved the *perceptual clarity* of the lower voice in a polyphonic texture (and of the text which it carried). In the early 18th century, 'consonance' and 'dissonance' came to be applied to *individual tones* in a chord, giving rise to a new interpretation of these terms which would eventually yield results in diametric opposition to all of the earlier forms of the CDC. Finally – in the mid-19th century – a conception of consonance and dissonance arose in which 'dissonance' was equated with "roughness," and this had implications quite different from those of earlier forms of the CDC.
>
> ... I ... suggest the following [terminology]: for CDC-1, monophonic or *melodic* consonance and dissonance; for CDC-2, *diaphonic* consonance and dissonance; for CDC-3, polyphonic or *contrapuntal* consonance and dissonance; for CDC-4, *triadic* consonance and dissonance (this form is often called "functional,", but this is not altogether accurate either, and might be better reserved for the more *purely* functional conception articulated by Riemann ... and finally – for CDC-5 – *timbral* consonance and dissonance.[50]

50. James Tenney, *A History of Consonance and Dissonance*, First Edition (UK: Routledge, 1988), p. 4 and p. 100.

The first four CDCs, numbered in order of their historical emergence, are essentially musical, derived from theory and practice; the most recent is psychoacoustic, proposed by Helmholtz.

The German philosopher and phenomenologist Carl Stumpf (1848–1946) offered an alternative perceptual theory to Helmholtz's categories of smoothness and roughness. His thesis was that consonance is determined by the extent to which an aggregate resembles a single tone: namely, by how well the partials match a single harmonic series.[51] This psychoacoustic reformulation of CDC-2 appears to be borne out by contemporary studies.

Composers Marc Sabat and Wolfgang von Schweinitz proposed a scale of *relative* consonance based on "tuneable intervals". Originally, these were established by empirical means. All of the ratios up to three octaves wide, generated from the first 28 harmonics, were tested, and any intervals that could not be directly tuned as dyads when played simultaneously in the middle register on string instruments were eliminated. The remaining intervals were placed in three broad groupings based on the difficulty with which they could be tuned. Notably, narrower intervals, which fall within a critical band – all intervals melodically smaller than $\frac{9}{8}$ – were determined to *not* be tuneable. The largest denominator which was found to produce a tuneable interval was 12. Sabat suggested an expanded definition of consonance, also based on CDC-2, to be *any tuneable sonority*.

The concept of tuneable intervals proposes a new *musical* definition of the terms consonance and dissonance, one that does not contextually prescribe or proscribe any sonority but simply *distinguishes* those sounds that may be *tuned exactly*. The definition may be made more precise (and quantifiable) according to the following model. Given any integers a and b in lowest terms, their ratio is said to make a potentially tuneable interval *tuneable* between two tones of appropriate timbral richness with fundamental frequencies f and $\frac{b}{a}f$ if the periodicity pitch $\frac{f}{a} \geq 20$ Hz and the least common partial $bf \leq 6000$ Hz. These (approximate) limits are based on empirical testing, and appear to closely match the frequency range (ca. 30–5000 Hz), within which the human auditory system demonstrates a capacity for neural encoding of temporal information. Additional size restraints may be placed based on desired prime limit, critical bandwidth (depending on f), and on maximum interval size, for example in the middle register $\frac{9}{8} \leq \frac{b}{a} \leq \frac{8}{1}$. Tuneable intervals may be ordered in terms of relative consonance, based on various measures discussed in the next section.

51. Tenney, p. 30.

8. Harmonic distance and intersection

Harmonic fusion is perceived as the melding of an aggregate of individual pitches accompanied by a distinctive vibrating sonority called periodic signature, discussed in Section 1. If the periodicity pitch falls in the sub-audio range (*ca.* beyond the lowest A of the piano), the phenomenon of acoustic beating between partials assumes a dominant role. As the frequency of the periodicity pitch rises into the lower part of the audio range and eventually passes through the register of the human speaking voice, fusion becomes increasingly tangible and different frequency ratios create very clear, distinct qualities of sound. In higher ranges, the contrast between consonance and dissonance becomes again less discernible as the rate of interference between frequencies increases.

Generally, smaller ratios result in smoother perceived fusion within a given register. The simplest ratios take the form $\frac{n}{1}$ and are called *absolute consonances*. They produce the smoothest manifestations of harmonic fusion because all partials of the upper pitch are potentially harmonics of the lower pitch. As intervals increase in numerical complexity, *i.e.* as the denominator increases, the periodicity pitch descends. Fusion takes on a periodic purring or roughness that is the most characteristic sonority associated with just intonation. Eventually, ratios become so complex that the fusion sound is blurred and no longer readily distinguishable from irrational tempered or mistuned intervals.

Tenney defined a measure called *harmonic distance*. Smaller harmonic distance values correspond to a smaller "city block" distance between pitches, as measured in *harmonic space*, which is defined as a multi-dimensional lattice with prime-number absolute consonances of the form $\frac{p}{1}$ as fundamental steps of each dimension.[52] For a ratio $\frac{b}{a}$ in lowest terms, its harmonic distance is calculated by the following expression.

$$HD = \log_2(b \times a) \tag{3}$$

The product of b and a is equivalent to the least common partial shared by harmonic series above b and a. In Equation (3), harmonic distance is, therefore, a measure of the number of octaves, evaluated exponentially, from the periodicity pitch (1) to the least common partial ($b \times a$). Since every rational number can be uniquely expressed as a product of powers (α) of prime numbers

52. Tenney, *From Scratch, Writings in Music Theory*, Chapter 12. Tenney thought of harmonic distance as a measure applicable to both successively and simultaneously sounded intervals. He believed that it could be usefully applied to pure tones (sinewaves) as well as spectrally rich timbres.

(p), the following expressions may be formulated.

$$\frac{b}{a} = \prod_i p_i^{\alpha_i} \quad \longrightarrow \quad b \times a = \prod_i p_i^{|\alpha_i|}$$

HD may, therefore, be expressed more specifically as a summation of powers of each prime factor, since $\log(m \times n) = \log m + \log n$ and $\log x^k = k \log x$.

$$HD = \sum_i |\alpha_i| \log_2 p_i \tag{4}$$

Equation (4) demonstrates the "city block" interpretation of harmonic distance.

The interval with the smallest harmonic distance is the unison, as $\log_2(1 \times 1)$ returns an *HD* value of 0. While it may sometimes seem counterintuitive to conceive of the unison as an "interval", the concept of harmonic distance affords an insight into this question. It is logical that two identical frequencies would have the *smallest* possible harmonic distance – that is, have *no* harmonic distance. This is because the partials of *either* note may be absorbed into the harmonic series of the *other*, colouring its timbre; the unison is the only interval with this symmetrical property.

It is often proposed that harmony and timbre are, in fact, equivalent. But in this case, it is clear that *harmony* is the more *general* principle: two different timbres, tuned in unison, have an harmonic distance of 0. In musical terms, this simply states that an harmonic construct may often retain its identity even when the timbres are altered. At the same time, harmony, particularly when tuned in just intonation, is actually dynamically altering timbres as well as creating new ones.

A closely related measure, also from Tenney, called *intersection*[53] (I) may be used to express the ratio between aggregate partials of two complex tones and the entire harmonic series of their periodicity pitch. Both measures offer similar, but interestingly differentiated rankings of *relative consonance*.

Given a ratio in lowest terms $\frac{b}{a}$, the intersection of the interval with respect to its periodicity pitch may be expressed as the following.

$$I = \frac{a + b - 1}{ab} \tag{5}$$

This equation counts the number of partials of each pitch a and b up to their least common partial ab subtracting 1 so that the least common partial itself is only counted once. These partials are, respectively, $\{a, 2a, 3a, ..., ba\}$ and $\{b, 2b, 3b, ..., ab\}$. Since the pattern of partials repeats between each multiple of ab, the least common partial also represents the interval's *harmonic period*.

53. Tenney, Chapter 11.

Unlike harmonic distance, which does not differentiate otonal and utonal structures, harmonic intersection does and, therefore, may be extended to aggregates of any number of pitches tuned in just intonation to provide an accurate quantitative measure of harmonicity.

9. Equal division tone systems

Equal-division systems, discussed earlier as a way of comparing the melodic size of intervals, are also used to compose music in their own right. Advantages of such tone systems include their various symmetric properties, including the ability to transpose by any interval, *i.e.* to duplicate identical relationships at any pitch-height without the need to introduce new pitches. They also offer the possibility of approximating rational intervals using a set of fixed pitches that serve as a *temperament*, or deliberate mistuning of the rational intervals. For instruments whose pitches are fixed, equal temperaments offer a practical tuning "compromise", possibly applicable to various compositions without requiring a complete retuning.

There are an infinite number of possible rational intervals, just as there are infinitely many pitch-classes in any harmonic series. A temperament represents these as best as possible with a finite set of options. A *consistent* temperament, as defined by Paul Ehrlich (b. 1972), is one in which the best representation of constituent intervals – up to some fixed limit – sum to the best representation of the combined interval.[54] Inconsistency – or "roundoff error", which any temperament eventually exhibits – results in inherent ambiguities, or paradoxes, arising between symmetrical properties of the tone system and the harmonic intervals it seems to represent. This ambiguity may or may not be considered a musically interesting quality. It is, however, important to note that it is based on *fooling the ear*.

In 12-ED2, certain formulaic "tonal progressions" – in particular cyclic sequences (by thirds, fifths, *etc.*) – are made possible without microtonal dissonances or transpositions, taking advantage of this system's "rough pixelation" or "soft focus". Strict JI composers like Johnston reject this approach as being a form of trickery or deception and, therefore, intellectually dishonest. Advocates of temperament, like Jean-Philippe Rameau (1683–1764), on the other hand, praise the magical quality of being transported to a new region

54. In this sense, 12-ED2 is consistent, for example, in a way that 24-ED2 is not. Consider the chord 4:5:7, which is comprised of a major third and a diminished fifth, outlining a natural seventh. In 12-ED2 the nearest representations of each interval, $\frac{5}{4} \approx 4$ steps and $\frac{7}{5} \approx 6$ steps, add up to the nearest representation of the outer interval, $\frac{7}{4} \approx 10$ steps. In 24-ED2 the respective values are 8 steps, 12 steps and 19 steps, which do not sum.

of the harmonic space (by an enharmonic reinterpretation of the diminished chord, for example), sensing "all the harshness"[55] of the quartertone without having it explicitly articulated as part of a dissonant interval.

Elaborated examples of composition using fine-grained equal-divisions may be found in the works of Carrillo and Wyschnegradsky. Drawing on his experience as a violinist, Carrillo developed a microtonal point of view about musical exploration, which he called "sonido 13" – the thirteenth sound. A talented musician, he was one of the first composers to write in a microtonal idiom for the symphony orchestra. Wyschnegradsky, on the other hand, composed a large body of microtonal music that was almost exclusively written for ensembles featuring variously retuned pianos, occasionally in combination with other instruments. An influential teacher and mentor, his theoretical writings, in particular *La Loi de la Pansonorité* from 1953, describe an ultrachromatic pitch-space that pixelates the perceived frequency range into a palette of compositionally accessible tonal material. His work inspired an entire generation of microtonal composers, including Bruce Mather (b. 1939) and Pascale Criton (b. 1954).

In such contexts, the interpretation of intervals no longer depends entirely *on their sound alone*, but must also be *deduced from their context*. This alteration of listening focus is perhaps the fundamental difference between conceiving of music in a temperament or in just intonation. The increased interest today in exploring perceptions of *sound* and *time* as fundamental materials of music invites a kind of listening, which the appreciation of just intonation also depends upon.

The most commonly explored ED2s may be loosely divided into three groups. First, there are numerous systems based on the antiprime 12, which may readily be divided in 2, 3, 4 and 6 equal units, and which generates various multiples (24, 36, 48, 72, 144, *etc.*). These systems share the well-known "closed circle" of 12 well approximated fifths,[56] replicating it at various microtonal offsets. 72- and 144-ED2 closely approximate some of the most common tuneable rational intervals up to the 23-limit, which has led to their adoption for quasi-JI works by some composers, including Ezra Sims (1928–2015), James Tenney, Hans Zender (b. 1936), and Georg Friedrich Haas (b. 1953), and Marc Sabat, among others.

Composers associated with the European "spectral" technique, pioneered

55. Deborah Hayes, "Rameau's theory of harmonic generation: an annotated translation and commentary of Génération Harmonique by Jean-Philippe Rameau" (PhD diss., Stanford University, 1968), p. 178.

56. The 12-ED2 fifth is almost 2 cents smaller than the JI ratio $\frac{3}{2}$, a difference that beats slowly in the middle register but is only minimally perceptible on the modern piano.

by Karlheinz Stockhausen (1928–2007), Horațiu Rădulescu (1942–2008), Gérard Grisey (1946–1998), Tristan Murail (b. 1947), and Claude Vivier (1948–1983), have often used various approximations to simulate the harmonic series, combination tones and distorted spectra. Rădulescu's music is especially noteworthy for its extensive exploration of *scordature* and use of extremely high natural harmonics.

Two additional groups of equal-division systems approximate meantone or Pythagorean tunings respectively. The "extended meantone" systems (see Table 2) divide the 5 wholetones within an octave into unequal parts, a smaller "chromatic" and a larger "diatonic" semitone. The "Pythagorean" systems (see Table 3) take the same number of units per wholetone but reverse the numbers assigned to the chromatic and diatonic semitones.

53-ED2, in particular, has received considerable attention. Its step size, 22.64 cents, is known as Mercator's comma (κ_M) and lies between the syntonic comma (κ_5) and the Pythagorean comma (κ_3). It was used to describe the Turkish tone system in a theory developed by Suphi Ezgi (1869–1962),[57] and was advocated as an ideal tuning by 19th century European theorists, considered harmonically superior to 12-ED2 because of its very close approximation of many consonant Ptolemaic ratios. In particular, it is the smallest ED2 to clearly approximate both the major and the minor wholetones.

Perhaps the most obvious point of objection to any equal-division system is the problem of *melodic granularity*, *i.e.* the degree to which very subtle differences between various JI intervals are tempered out. One of the objections to Suphi Ezgi's comma-based approximation of Turkish intervals is that the very subtle contextual variations of intonation in melody are not accurately represented. Similarly, in a harmonic context, voices and instruments of flexible pitch make very subtle adjustments to distinguish particular progressions. The only way to achieve this kind of accuracy, in fact, is to define each interval exactly, within the bounds of auditory perception. Only a just intonation tuning system or an equal-division system finer than the JND offer possibilities of realising such an approach.

57. Ioannis Zannos, *Ichos und Makam: Vergleichende Untersuchungen zum Tonsystem der griechisch-orthodoxen Kirchenmusik und der türkischen Kunstmusik* (Bonn: Orpheus Verlag, 1994), p. 18.

Table 2: Comparison of some *quasi-meantone* equal-division systems.

Number of divisions	Size of fifths in cents	Units per wholetone	Size in cents	Units per diatonic semitone	Size in cents
19-ED2	694.74 $\left(\frac{1}{3}\kappa_5\right)$	3	189.47	2	126.32
31-ED2	696.77 $\left(\frac{1}{4}\kappa_5\right)$	5	193.55	3	116.13
43-ED2	697.67 $\left(\frac{1}{5}\kappa_5\right)$	7	195.35	4	111.63
55-ED2	698.18 $\left(\frac{1}{6}\kappa_3\right)$	9	196.36	5	109.09

Table 3: Comparison of some *quasi-Pythagorean* equal-division systems.

Number of divisions	Size of fifths in cents	Units per wholetone	Size in cents	Units per diatonic semitone	Size in cents
17-ED2	705.88	3	211.76	1	70.59
29-ED2	703.45	5	206.90	2	82.76
41-ED2	702.44	7	204.88	3	87.80
53-ED2	701.89	9	203.77	4	90.57

10. Toward a possible music

It is true that the theory and mechanisms of just intonation are fascinating in their own right and, as many tuning enthusiasts would agree, necessary to confront. To truly appreciate the enormous musical potential of rational tuning as composers, interpreters, and, most importantly, as listeners, it is crucial to be immersed in it *practically* in order to connect theory with sound. After all, just intonation does not require theoretical minutiae – as elegant as they may be – to justify its beauty and musical value; it is already intuitively ingrained in the way humans hear. An understanding of ratios, cents, commas, *etc.* merely provides tools to better comprehend and compare what is heard (*śruti*) as well as to imagine, invent, and develop what *could* be heard.

11. Score excerpts

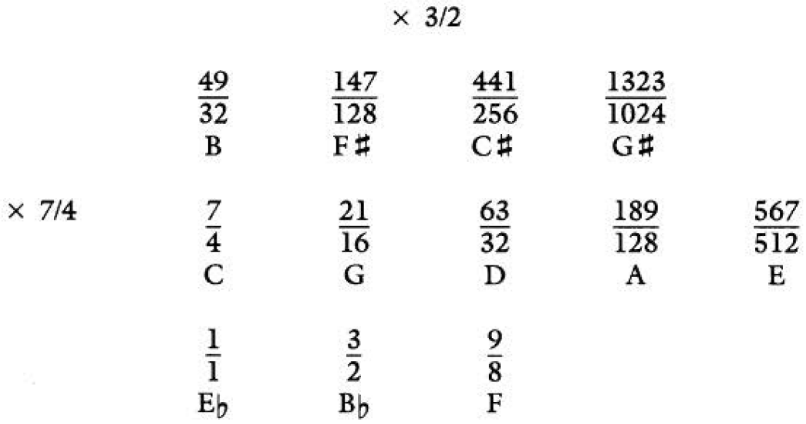
Figure 1: 5-limit enharmonic intervals as in bar 4 of *Tenebrae factae sunt* (1611) by Carlo Gesualdo di Venosa.

Figure 2: Excerpt from an 11-limit trio published by Giovanni Battista Doni, composer unknown. *Compositione per il Diatonico Equabile* (ca. 1637) transcribed into *HEJI*.

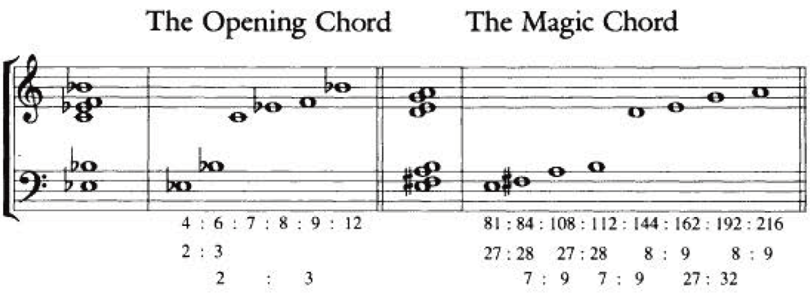
Figure 3: 7-limit "enharmonic meantone" excerpt from *Toccata Settima* (ca. 1640) by Michelangelo Rossi.

Figure 4: 7-limit excerpt from *By the Rivers of Babylon* (1931) by Harry Partch.

Figure 5: Three examples from Kyle Gann's analysis and transcription of La Monte Young's *The Well Tuned Piano* (1964–73–81–present).



EXAMPLE 2



EXAMPLE 10

EXAMPLE 14

Figure 6: Horațiu Rădulescu's Fourth String Quartet *"infinite to be cannot be infinite, infinite anti-be could be infinite"* (1976–1987) for nine string quartets with spectral scordatura.

Figure 7: 13-limit excerpt from Ben Johnston's *String Quartet No. 5* (1979) with *otonal* and *utonal* chords in extended just intonation.

Figure 8: 7-limit excerpt from James Tenney's *Harmonium No. 7* (2000) with approximate arrow and exact ratio notation.

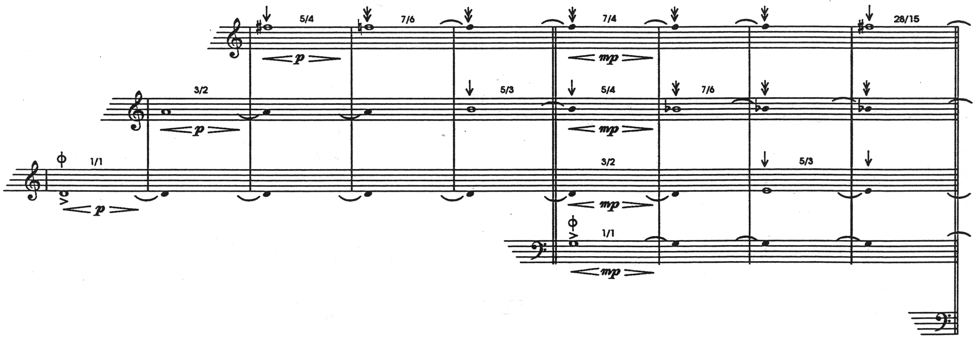


Figure 9: 13-limit excerpt from *DIE KANTATE oder, Gottes Augenstern bist du* (2002–3) for speaking voice, soprano, violin, viola, horn, tuba and live sound projection by Wolfgang von Schweinitz.

Figure 10: 13-limit excerpt from *Gradients of Detail* (2006) by Chiyoko Szlavnics.

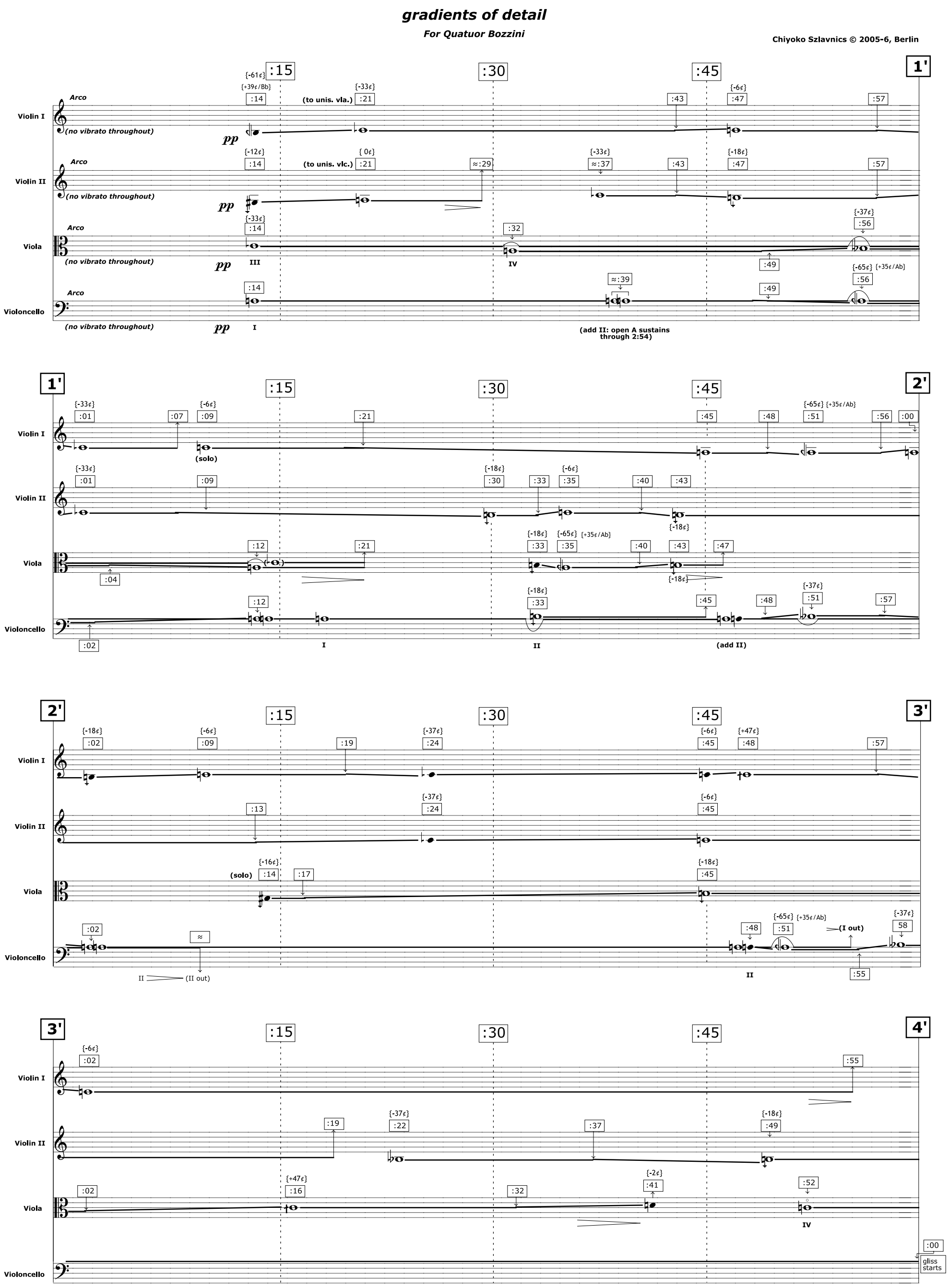


Figure 11: 11-limit excerpt from *Asking ocean* (2016) for solo string quartet and 16 instruments by Marc Sabat.

Figure 12: 7-limit excerpt from *Prisma Interius V* (2017) by Catherine Lamb.

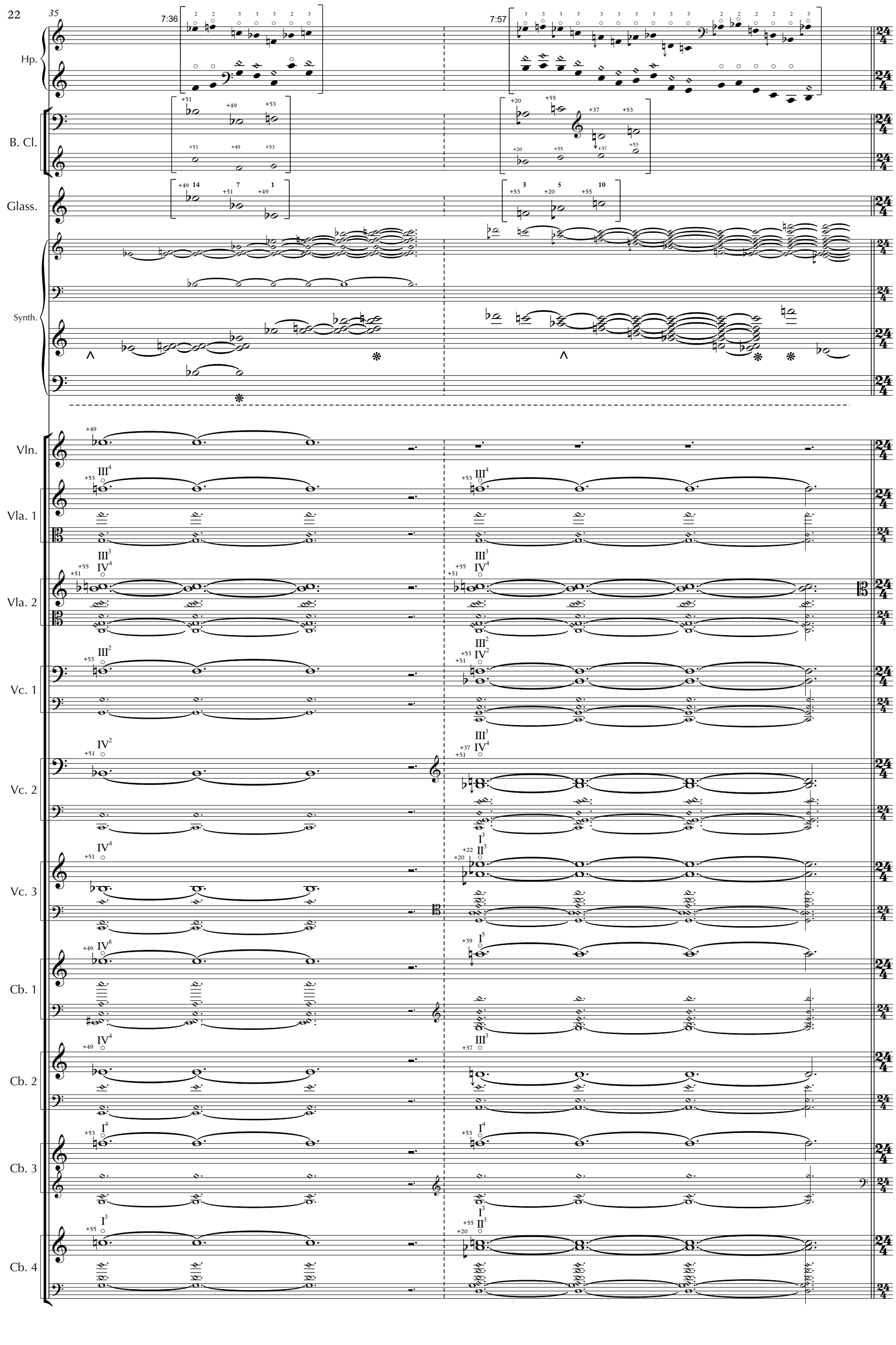
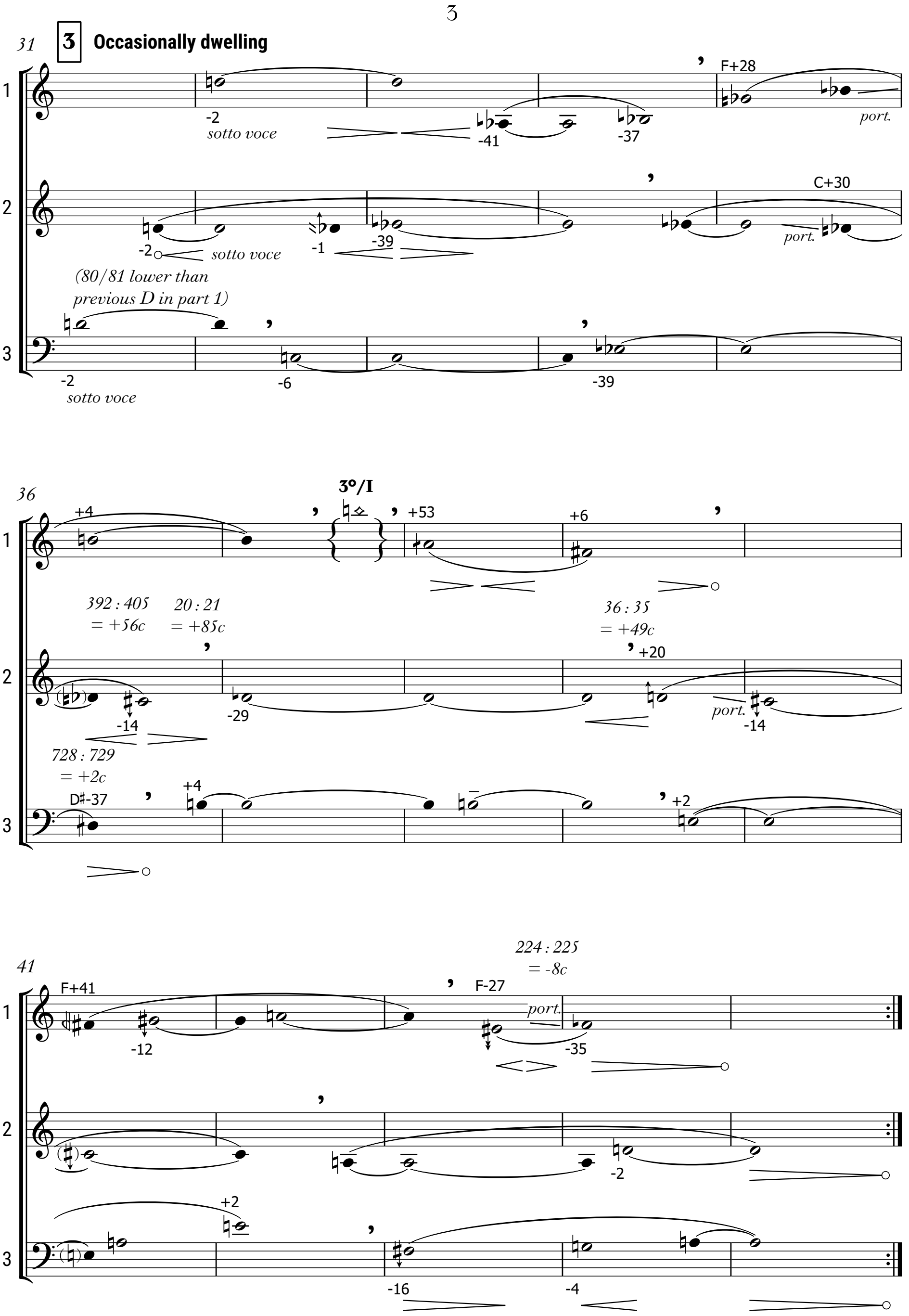


Figure 13: 17-limit excerpt from *BRANCH: Plainsound Trio* (2018) for three sustaining instruments or voices by Thomas Nicholson.



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References

Barbieri, Patrizio. *Enharmonic Instruments and Music 1470–1900*. Rome: Il Levante Libreria Editrice, 2008.

Barbour, James Murray. *Tuning and temperament: a historical survey*. Chicago: Michigan State University Press, 1951.

Barker, Andrew. *Greek Musical Writings, Volume II: Harmonic and Acoustic Theory*. Cambridge: Cambridge University Press, 1997.

Berger, Karol. *Musica ficta: Theories of accidental inflections in vocal polyphony from Marchetto da Padova to Gioseffo Zarlino*. Cambridge: Cambridge University Press, 2004.

Blackwood, Easley. *The Structure of Recognizable Diatonic Tunings*. Princeton: Princeton University Press, 1986.

Boethius, Anicius Manlius Severinus. *De institutione musica*. Edited by Claude V. Palisca. Translated by Calvin M. Bower. New Haven: Yale University Press, 1989.

Cowell, Henry. *New Musical Resources*. New York: Alfred A. Knopf, 1930.

Deutsch, Diana. *The Psychology of Music*. Third Edition. San Diego: Elsevier, 2013.

Fechner, Gustav Theodor. *Elements of psychophysics [Elemente der Psychophysik]*. Edited by D. H. Howes and E. G. Boring. Translated by H. E. Adler. Vol. 1. New York: Holt, Rinehart / Winston, 1966.

Gann, Kyle. "La Monte Young's The Well-Tuned Piano." *Perspectives of New Music* Volume 31, no. 1 (Winter 1993): 134–162.

Hayes, Deborah. "Rameau's theory of harmonic generation: an annotated translation and commentary of Génération Harmonique by Jean-Philippe Rameau." PhD diss., Stanford University, 1968.

Helmholtz, Hermann von. *On the Sensations of Tone as a Physiological Basis for the Theory of Music*. Second English Edition. Translated by Alexander J. Ellis. New York: Dover, 1954.

Johnston, Ben. *Maximum Clarity and Other Writings on Music*. Edited by Bob Gilmore. Chicago: University of Illinois Press, 2006.

Lang, Klaus. *Auf Wohlklangswellen durch der Töne Meer: Temperaturen und Stimmungen zwischen dem 11. und 19. Jahrhundert*. Graz: Institut für Elektronische Musik (IEM) an der Universität für Musik und darstellende Kunst in Graz, 1999.

Lusitano, Vicente. *Introduttione facilissima, et novissima, di canto fermo, figurato, contraponto semplice, et in concerto*. Rome: Antonio Blado, 1553.

midi.org. *The MIDI 1.0 Specification*. https://www.midi.org/specifications-old/item/the-midi-1-0-specification, 1982.

Partch, Harry. *Genesis of a Music*. Second Edition. Cambridge: Da Capo Press, 1979.

Ramos de Pareja, Bartolomé. *Musica practica*. ?Bologna: Baltasar de Hiriberia, 1482.

Sabat, Marc. *The Extended Helmholtz-Ellis JI Pitch Notation*. http://www.marcsabat.com/pdfs/notation.pdf, 2005.

Schweinitz, Wolfgang von. *The Classical Indian Just Intonation Tuning System notated in the Extended Helmholtz-Ellis JI Pitch Notation (2007) — with a commentary by musicologist Prof. P. Sambamurthy copied from his great anthology SOUTH INDIAN MUSIC*. http://www.plainsound.org/pdfs/srutis.pdf, 2006.

Secor, George, and David Keenan. *Sagittal: A Microtonal Notation System*. http://sagittal.org/sagittal.pdf, 2012.

Smith, Charles Samuel. “Leonhard Euler’s Tentamen novae theoriae musicae : a translation and commentary.” PhD diss., Indiana University, 1960.

Tartini, Giuseppe. *Trattato di musica: secondo la vera scienza dell’armonia*. Padua: Giovanni Manfrè, 1754.

Tenney, James. *A History of Consonance and Dissonance*. First Edition. UK: Routledge, 1988.

———. *From Scratch, Writings in Music Theory*. Edited by Larry Polansky, Lauren Pratt, Robert Wannamaker, and Michael Winter. Illinois: University of Illinois Press, 2015.

Vicentino, Nicola. *L’antica musica ridotta alla moderna prattica*. Venice: Antonio Barre, 1555.

Wyschnegradsky, Ivan. *La Loi de la Pansonorité*. Edited by Franck Jedrzejewski and Pascal Criton. Paris: Contrechamps, 1996.

Zannos, Ioannis. *Ichos und Makam: Vergleichende Untersuchungen zum Tonsystem der griechisch-orthodoxen Kirchenmusik und der türkischen Kunstmusik*. Bonn: Orpheus Verlag, 1994.

Zarlino, Gioseffo. *Le istitutioni harmoniche*. Venice, 1558.